Clearly (ΣB) $v = \bar{x}$ holds. Set $A = \mathrm{supp}(x) \setminus B$ and $\Gamma = \mathrm{supp}(\bar{x})$. Obviously, A, B and Γ are fixed under G_x ; we show that G_x acts transitively on A, B and Γ. The elements of order 7 in G have exactly 2 fixed points in Ω . Since $|G_x|$ does not divide $|M_{22}|$, G_x has no fixed point in Ω . From $7 \mid |G_x|$ we readily conclude that G_x acts 2-transitively on A and that G_x permutes transitively the 7 4-sets in Φ_{49} which are contained in B. Let u be a 4-set in Φ_{49} contained in B. Since v is a G-homomorphism $G_u \leq G_{uv} = G_{\bar{x}} = G_x$. So from (3.8) it follows that G_x acts transitively on B.

Let $G_{\dot{u}}$ denote the largest subgroup of G_u fixing every vertex in u. From the fundamental properties of the Higman-Sims graph it follows that $G_{\dot{u}}$ has exactly 4 orbits of length 16 in Γ which are conjugate under G_u by (3.8). Hence G_u acts transitively on Γ . Since G_x acts doubly-transitively on A, the Higman-Sims graph induces on A the null graph. Moreover, the preceding discussion shows that for $\beta \in B$ we have $|\Delta(\beta) \cap B| = 2$ and $|\Delta(\beta) \cap \Gamma| = 16$. It easily follows that the matrix of the Higman-Sims graph with respect to G_x is as asserted; in particular $A = \Lambda_{14}(x)$, $B = \Lambda_6(x)$ and $\Gamma = \Lambda_8(x)$. □

(3.10) PROPOSITION.

(1) $w_i(H_{22}) = 0$ for $0 < i < 32$ and $i \neq 22,30$.

(2) $w_{22}(H_{22}) = 100$ and $W_{30}(H_{22}) = 1.100$.

(3) $W_{22}(H_{22}) = \{\Delta(\gamma) \mid \gamma \in \Omega\}$ is a G-orbit; $G_{\Delta(\gamma)} = G_\gamma$ for $\gamma \in \Omega$.

(4) $W_{30}(H_{22})$ is a G-orbit. For $x \in W_{30}(H_{22})$ we have $\Omega : G_x = \{\Lambda_8(x), \Lambda_6(x)\}$ where $\mathrm{supp}(x) = \Lambda_8(x)$, $\mathrm{supp}(x+\mathbb{1}) = \Lambda_6(x)$. In particular $\lambda_8(x) = 30$ and $\lambda_6(x) = 70$.

<u>Proof.</u> $H_{22} \leq H_{99} = \langle \mathbb{1} \rangle^\perp$ implies that $w_i(H_{22}) = 0$ for every odd i. Let $0 \neq x \in H_{22}$ such that $w(x) < 32$. From (3.5) it follows that either $w(x) = 30$ or $w(x) \leq 26$. Set $\lambda_j = \lambda_j(x)$.

(i) Suppose that $w(x) \neq 30$. We claim that in this case $w(x) = 22$ and $x = \Delta(\gamma)$ for some $\gamma \in \Omega$.

- Assume the contrary. We may choose x as a counterexample of minimal weight. If $\lambda_{22} \neq 0$ there existed a $\gamma \in \Lambda_{22}(x)$, and then $w(x) \geq 22$ and $w(x+\Delta(\gamma)) \leq 4$, consequently $x+\Delta(\gamma) = 0$ and $x = \Delta(\gamma)$ by (3.3), contrary to the assumption.

If $\lambda_0 \neq 0$ we had also $\lambda_{22} \neq 0$ by (3.6). Hence we may suppose that $\lambda_{22} = 0 = \lambda_0$.

We have $w(x+\Delta(\beta)) \geq w(x)$ for all $\beta \in \Omega$. For, otherwise, $x+\Delta(\beta) = \Delta(\gamma)$ for some $\gamma \in \Omega$ by the minimal choice of x, so $x = \Delta(\beta) + \Delta(\gamma) = (\beta+\gamma)v$ which contradicts (3.5) because of $\beta+\gamma \in \Phi_{21} \cup \Phi_{22}$.

For $\beta \in \Lambda_j(x)$ we therefore have $w(x) + 22 - 2j \geq w(x)$ and $j \leq 11$ follows. Therefore $\lambda_j \neq 0$ implies $j \in \{6,8,10\}$ by (3.7).

Counting the edges of the Higman-Sims graph between supp(x) and now yields the equations

$$22\, w(x) = 6\lambda_6 + 8\lambda_8 + 10\lambda_{10}$$

$$100 = \lambda_6 + \lambda_8 + \lambda_{10} .$$

From these equations we deduce $11\, w(x) = 300 + \lambda_8 + 2\lambda_{10} \geq 300$ and $w(x) > 27$, again a contradiction.

So the claim is proved and (1) holds. We also have obtained that $W_{22}(H_{22})$ is precisely the set of all adjacency vectors $\Delta(\beta)$, $\beta \in \Omega$. Hence (3) holds and $w_{22}(H_{22}) = 100$.

(ii) Suppose $w(x) = 30$. Our first claim is that $\lambda_8 = 30$ and $\lambda_6 = 70$.

- Since $w_8(H_{22}) = 0$ by (1), we have $\lambda_{22} = 0$. From (3.6) we infer that $\lambda_0 = 0$. Furthermore, for all $\beta \in \Omega$ $w(x+\Delta(\beta)) \geq 30$, since otherwise $x + \Delta(\beta) = \Delta(\gamma)$ for some $\gamma \in \Omega$ and $x \in H_{21}$ which is impossible.

Therefore we have again $\lambda_j \neq 0$ only possibly for $j \in \{6,8,10\}$ by (3.7). We assert that also $\lambda_{10} = 0$.

For, if $\beta \in \Lambda_{10}(x)$ we had $w(x+\Delta(\beta)) = 32$, hence $x + \Delta(\beta) = \Delta(\gamma) + \Delta(\delta)$ for some $\gamma, \delta \in \Omega$ by (3.5) and (3.2), which entails that

$$x = \Delta(\beta) + \Delta(\gamma) + \Delta(\delta) = (\beta+\gamma+\delta)v ,$$

a contradiction against (3.2). Counting edges yields the equations

$$6\lambda_6 + 8\lambda_8 = 22\cdot 30 = 660$$
$$\lambda_6 + \lambda_8 = 100$$

which have the unique solution $\lambda_6 = 70$ and $\lambda_8 = 30$.

(iii) For every $x \in W_{30}(H_{22})$ and for every $\beta \in \Lambda_8(x)$ we have $x + \Delta(\beta) \in W_{36}(H_{22}) = W_{36}(H_{21})$ which is a G-orbit of length 4.125. On the other hand for every $z \in W_{36}(H_{22})$ and every $\gamma \in \Lambda_{14}(z)$ we have $z + \Delta(\gamma) \in W_{30}(H_{22})$. Hence $W_{30}(H_{22}) \neq 0$.
We consider the incidence structure

$$\mathfrak{J} = (W_{30}(H_{22}), W_{36}(H_{22}), I)$$

where $I = \{(x,z) \mid x \in W_{30}(H_{22}),\ z \in W_{36}(H_{22}) \text{ and } x + z \in W_{22}(H_{22})\}$.
It follows from (3.9) that G acts transitively on I. Since G acts transitively on $W_{36}(H_{21})$ we conclude that also $W_{30}(H_{22})$ is a G-orbit and that G_x acts transitively on $\Lambda_8(x)$ for $x \in W_{39}(H_{22})$.
Double counting of $|I|$ gives

$$w_{30}(H_{22}) \cdot 30 = |I| = w_{36}(H_{22}) \cdot 8 = 4.125 \cdot 8 = 33.000,$$

hence $w_{30}(H_{22}) = 1.100$. From (3.9) we may deduce that for $x \in W_{30}(H_{22})$ that $\Lambda_8(x) \cap \mathrm{supp}(x) = \emptyset$, hence $\Lambda_8(x) = \mathrm{supp}(x)$ and $\Lambda_6(x) = \mathrm{supp}(x+\mathbb{1})$. One checks by inspection that for $z \in W_{36}(H_{22})$ and $\beta \in \Lambda_{14}(z)$ $G_{z,\beta}$ has an orbit of length 14 in $\mathrm{supp}(z+\Delta(\beta) + \mathbb{1})$. It again follows by counting of edges that for $x \in W_{30}(H_{22})$, G_x acts transitively on $\mathrm{supp}(x+\mathbb{1}) = \Lambda_6(x)$. □

Remark. It will be shown in (4.9) that $G_x \cong \Sigma_8$ for $x \in W_{30}(H_{22})$.

The results we have yet obtained are sufficient to determine the weight distribution of the code H_{22} via the MacWilliams identities.

(3.11) THEOREM. The weight districution of H_{22} and the orbits of G in H_{22} are as described in the following table.

i	$w_i(H_{22})$	length of G-orbits in $W_i(H_{22})$
0/100	1	1
22/78	100	100
30/70	1.100	1.100
32/68	3.850	3.850
36/64	4.125	4.125
38/62	38.500	38.500
40/60	92.400	15.400 , 77.000
42/58	193.600	61.600 , 132.000
44/56	347.600	1.100 , 346.500
46/54	485.100	23.100 , 462.000
48/52	600.600	231.000 , 369.600
50	660.352	44.352 , 616.000.

In particular G has precisley 34 orbits in H_{22}. Complementary vectors of weight 50 are in the same G-orbit.

<u>Proof.</u> Let $a_i = w_i(H_{22})$ and $b_i = w_i(H_{78})$. We recall that $H_{78} = H_{22}^{\perp}$. Hence the families (a_i) and (b_i) are related to each other by the Mac Williams identities.
We have $H_{22} \leqslant H_{78} \leqslant H_{99}$; hence for every odd i $a_i = b_i = 0$ holds. Since $\mathbb{1} \in H_{22}$, we have $a_i = a_{100-i}$ for all i. From (3.10) we have the information that

$$a_i = 0 \quad \text{for} \quad 0 < i < 32 \quad \text{and} \quad i \neq 22,30$$
$$\text{and } a_{22} = 100,\ a_{30} = 1.100.$$

Furthermore $a_0 = a_{100} = 1$ trivially holds, and from (3.5) every a_i with $i \equiv 0 \pmod 4$ is known. So the only unknown weight numbers of H_{22} with $i \leqslant 50$ are

$$a_{34},\ a_{38},\ a_{42},\ a_{46} \text{ and } a_{50} .$$

On the other hand we know from (3.3) and (3.4) the values

$$b_0 = 1$$
$$b_2 = b_4 = 0$$
$$b_6 = 3.850$$
$$b_8 = 119.625 .$$

It follows from the general theory of Mac Williams identities that the weight distribution (a_i) is uniquely determined by the known values of the a_i and b_j, see e.g. [21, Theorem]. Explicit calculations yield the values asserted in the Theorem.

It remains to determine the G-orbits in $H_{22} \setminus H_{21}$. It is convenient to consider first the factor module H_{22}/H_1 which consists of pairs $\{x, x+\mathbb{1}\} = x+H_1$ of complementary vectors, $x \in H_{22}$.
We clearly have $H_{22}/H_1 \underset{FG}{\cong} H_{100}/H_{79} \underset{FG}{\cong} H_{21}^{*}$ because of $H_{79} = H_{21}^{\perp}$.

Since $F = \mathbb{F}_2$, G has by Lemma (1.2) the same number of orbits in H_{21}^{*} and in H_{21}. By (3.5) therefore G has exactly 18 orbits in H_{22}/H_1. By (3.5) G also has 9 orbits in H_{21}/H_1, so G has exactly 9 orbits in $H_{22}/H_1 \setminus H_{21}/H_1$.
For $x+H_1 \in F\Omega/H_1$ we define the weight $w(x+H_1) = \{w(x), w(x+\mathbb{1})\}$.
From (3.10) it follows that G has in H_{22}/H_1 one orbit of elements of weight $\{22,78\}$ and length 100 and one orbit of elements of weight $\{30,70\}$ and length 1.100.
From (3.1), (3.2) and (3.3) it follows that G has in H_{22}/H_1 an orbit of elements of weight $\{38,62\}$ and length 38.500.
The weight distribution of H_{22} which we have determined tells us that all remaining pairs $x+H_1$ have weight $\{i,100-i\}$ where $i \in \{42,46,50\}$.
From (3.1), (3.2) and (3.3) we also obtain that G has in H_{22}/H_1 an orbit of elements of weight $\{42,58\}$ and length 61.600 and an orbit of elements of weight $\{46,54\}$ and length 23.100.
Since $a_{50}/2 = 330.176$ is not a divisor of the order of G we have at least 2 G-orbits of elements of weight $\{50,50\}$. Since G has exactly 9 orbits in $H_{22}/H_1 \setminus H_{21}/H_1$ it follows that G has precisely two more orbits: one of length 132.000 consisting of elements of weight $\{42,58\}$ and one of length 462.000 consisting of elements of weight $\{46,54\}$. To complete the proof of the Theorem we have to consider vectors of weight 50. We claim that there are even only two G-orbits in $W_{50}(H_{22})$, i.e. that complementary vectors in $W_{50}(H_{22})$ are in the same G-orbit. This assertion and the length of the two G-orbits in $W_{50}(H_{22})$ are obtained in the following Lemma. □

(3.12) LEMMA. Let P,Q be two distinct elements of $\Delta(\alpha)$ and $h, k, \ell \in \Delta_2(\alpha)$ such that $Q \in h$, $Q \notin k \cup \ell$, $P \notin h$, $h \cap k = \emptyset$ and $h \cap \ell \neq \emptyset$. Set $u = \alpha + P + Q + h$, $u_1 = u + k$ and $u_2 = u + \ell$ and let 50_i denote the G-orbit containing $x_i = u_i v$ $(i=1,2)$.
Then the following hold:

(1) $W_{50}(H_{22}) = 50_1 \cup 50_2$.

(2) $|50_1| = 44.352$ and $|50_2| = 616.000$.

Proof. Set $\bar{x} = uv$ and $x = \bar{x} + \mathbb{1}$. By construction $u \in \Phi_{47}$ and from (3.2) it follows that $x \in 48_2$, where 48_2 denote the G-orbit in $W_{48}(H_{22})$ of length 369.600. Furthermore we have $G_x = G_{\bar{x}} = G_u \cong \Sigma_5$ (by considering a hexad stabilizer). Elementary considerations show that $\Lambda_{10}(x)$ splits into a unique G_x-orbit Ψ_1 of length 6 and its complement Ψ_2 of cardinality 30. Moreover $k \in \Psi_1$ and $\ell \in \Psi_2$.
It is easy to show that G_{x_1} acts transitively on $supp(x_1) = \Lambda_{12}(x_1)$. We therefore can count the incidences in the incidence structure

$$\mathfrak{I}_1 = (48_2, 50_1, I_1) \quad \text{where} \quad I_1 = \{(w,z) \mid w \in 48_2,\ z \in 50_1 \text{ and } w + z = \Delta(\gamma) \text{ such that } |\gamma G_w| = 6\},$$

obtaining $369.600 \cdot 6 = |50_1| \cdot 50$, hence $|50_1| = 44.352$.
(It easily follows that G_{x_1} also acts transitively on $supp(\bar{x}_1) = \Lambda_{10}(x_1)$ where $\bar{x}_1 = x_1 + \mathbb{1}$ and that x_1 and $\bar{x}_1$ are in the same G-orbit.)
The preceding arguments also show that $50_2 \neq 50_1$. Since G has exactly 2 orbits in $W_{50}(H_{22})/H_1$ we have either $|50_2| = 616.000$ or $|50_2| = 308.000$. In order to decide this question we consider $\Lambda_{16}(x_2)$ and we find $|\Lambda_{16}(x_2)| = 2$ by direct examination. If $\gamma \in \Lambda_{16}(x_2)$, then $x_2 + \Delta(\gamma) \in W_{40}(H_{22}) = 40_1 \cup 40_2$ where 40_1 denotes the G-orbit of length 15.400 and 40_2 denotes the G-orbit of length 77.000, see (3.11). From (3.5) we infer that for $y \in 40_2$ $\Lambda_6(y)$ is a G-orbit of length 16 and that $\Lambda_6(z) = \emptyset$ for any $z \in 40_1$.

Therefore we consider the incidence structure

$$\mathfrak{I}_2 = (40_1, 50_2, I_2), \quad \text{where}$$
$$I_2 = \{(w,z) \mid w \in 40_1,\ z \in 50_2 \text{ and } w + \Delta(\gamma) = z \text{ for some } \gamma\}.$$

Counting incidences gives $|40_1| \cdot 16 = |50_2| \cdot 2$,
hence $|50_2| = |40_1| \cdot 8 = 616.000$ which completes the proof of the lemma. □

The argumentation in (3.12) (and in (3.9) before) may be extended to all G-orbits in H_{22} in order to obtain results concerning the "set connectivity" in the Higman-Sims graph. We list the results in the following proposition, omitting proofs which tend to be tedious.

G-orbits consisting of vectors of weight k are denoted by k or k_1, k_2 using the convention that $|k_1| < |k_2|$.
We give the relevant information in the graph matrix of the G-orbits in H_{22} which is defined as follows:
The row and column indices are the G-orbits in H_{22}. The entry $m;n$ belonging to the ordered pair (k,ℓ) tells that there are exactly m vertices $\beta \in \Omega$ such that $x + \Delta(\beta) \in \ell$ for any $x \in k$ and that there are exactly n vertices $\gamma \in \Omega$ such that $y + \Delta(\gamma) \in k$. The entry $0;0$ is replaced by —.
We give the graph matrix in a reduced form from which the complete matrix is easily derived by considering complementary vectors.

(3.13) PROPOSITION. The graph matrix of the G-orbits in H_{22} is given by the following submatrix.

	22	30	38	42_1	42_2	46_1	46_2	50_1	50_2
0	100;1	–	–	–	–	–	–	–	–
32	2;77	–	60;6	32;2	–	–	–	–	–
68	–	–	–	–	–	6;1	–	–	–
36	–	8;30	–	–	64;2	28;5	–	–	–
40_1	–	–	–	40;10	–	–	60;2	–	–
40_2	–	1;70	16;32	–	24;14	6;20	36;6	–	16;2
60_2	–	–	1;2	–	–	–	–	–	16;2
44_1	2;22	–	–	56;1	–	–	–	–	–
56_1	–	–	–	–	–	42;2	–	–	–
44_2	–	–	4;36	8;45	16;42	–	32;24	–	32;18
56_2	–	–	–	–	–	–	8;6	–	32;18
48_1	–	–	4;24	8;30	–	–	32;16	–	32;12
52_1	–	–	–	–	8;14	4;40	12;6	–	32;12
48_2	–	–	–	–	10;28	2;32	30;24	6;50	30;18
52_2	–	–	–	2;12	–	–	20;16	6;50	30;18

Proof. Omitted. □

From (3.13) (or other elementary considerations) we can derive for any G-orbit k the minimum weight $m(k)$ of a vector u such that $uv \in k$.

(3.14) COROLLARY. The values $m(k)$ for the G-orbits k in H_{22} are as given in the following table.

k	$m(k)$	k	$m(k)$
0	0	100	8
32	2	68	6
36	6	64	4
40_1	4	60_1	4
40_2	4	60_2	4
44_1	2	56_1	6
44_2	4	56_2	6
48_1	4	52_1	6
48_2	6	52_2	4
22	1	78	7
30	5	70	5
38	3	62	5
42_1	3	58_1	7
42_2	5	58_2	5
46_1	5	54_1	3
46_2	5	54_2	5
50_1	5	50_2	5

Moreover, every $x \in F\Omega$ is congruent to a vector of weight at most 8 modulo H_{78} and to a vector of weight at most 5 modulo H_{79}. Hence every coset leader of H_{78} has weight at most 8. □

We may now use the Mac Williams transformation to obtain the weight distributation of the codes $H_{78} = H_{22}^{\perp}$ and $H_{79} = H_{21}^{\perp}$. (The computation has been carried out at the Rechenzentrum der Universität Tübingen by F.H. Florian using an ALDES program for computing with large numbers.)

(3.15) PROPOSITION. The weight distributions of H_{78} and H_{79} are as given in the following tables.

i	$w_i(H_{78})$
0/100	1
6/94	3 850
8/92	119 625
10/90	8 625 540
12/88	504 741 475
14/86	21 060 732 550
16/84	641 604 305 375
18/82	14 622 264 133 400
20/80	255 578 801 503 795
22/78	3 496 197 414 021 950
24/76	38 040 184 865 580 975
26/74	333 583 288 959 605 300
28/72	2 383 620 258 950 558 925
30/70	14 005 822 677 643 540 370
32/68	68 193 674 451 079 227 050
34/66	276 907 651 030 419 444 000
36/64	942 804 612 331 379 390 725
38/62	2 703 690 041 528 811 696 900
40/60	6 554 715 235 199 646 035 290
42/58	13 474 850 115 575 617 584 200
44/56	23 545 377 618 939 915 393 150
46/54	35 033 702 002 644 035 359 900
48/52	44 444 350 668 327 576 562 750
50	48 108 741 996 656 177 342 352

i	$w_i(H_{79})$
0/100	1
6/94	3 850
8/92	154 825
10/90	16 387 140
12/88	1 003 835 875
14/86	42 133 634 950
16/84	1 283 480 881 375
18/82	29 244 271 163 800
20/80	511 152 645 567 795
22/78	6 992 403 401 202 750
24/76	76 080 408 035 945 775
26/74	667 166 480 352 256 500
28/72	4 767 240 353 068 238 925
30/70	28 011 646 019 961 809 810
32/68	136 387 349 145 968 724 650
34/66	553 815 299 356 210 253 600
36/64	1 885 609 223 857 102 552 325
38/62	5 407 380 102 022 140 311 300
40/60	13 109 430 428 316 832 071 770
42/58	26 949 700 280 870 672 877 000
44/56	47 090 755 183 602 450 042 750
46/54	70 067 404 105 299 389 919 900
48/52	88 888 701 152 233 082 111 550
50	96 217 484 223 130 147 456 656

□

Our next purpose is to determine the weight distribution and the G-orbit structure of the codes H'_{22} and H''_{22}. These codes are conjugate under $\overline{G} = Aut(G)$; hence they have the same weight distribution and the same G-orbit structure so that they can be discussed simultaneously. Since $H_{23} = H_{22} \cup H'_{22} \cup H''_{22}$ we obtain all information also about H_{23}.

At first, we complete the classification of vectors in $W_8(H_{79})$.

(3.16) LEMMA. For every $x \in H_{79} \setminus H_{78}$ and every $\beta \in \Omega$
$w(x\,\Delta(\beta)) = |supp(x) \cap \Delta(\beta)|$ is odd.

Proof. This is an immediate consequence of (2.7)(8). □

(3.17) PROPOSITION. $W_8(H_{79}) \setminus W_8(H_{78})$ consists precisely of all vectors $\beta + m$ where m is a β-heptad.
In particular

$$w_8(H_{79}) = w_8(H_{78}) + 35.200$$

and G has exactly two orbits Φ'_8 and Φ''_8 of length 17.600 in $W_8(H_{79}) \setminus W_8(H_{78})$. These G-orbits are conjugate under $\overline{G} = Aut(G)$.
Choosing suitable notation we have $\Phi'_8 \subseteq H'_{78}$ and $\Phi''_8 \subseteq H''_{78}$.

Proof. Let m be a β-heptad, $\beta \in \Omega$. Then we have $(\beta+m)v = 1$, as follows from (1.9). Hence $\beta + m \in W_8(H_{79}) \setminus H_{78}$. From (1.3) we infer that $G_\alpha = M_{22}$ has two orbits on the set of α-heptads, the orbits being conjugate under the action of $\overline{G}_\alpha = Aut(M_{22})$. It follows that G has two orbits Φ'_8 and Φ''_8 of length 17.600 in $W_8(H_{79}) \setminus W_8(H_{78})$ which are conjugate under $\overline{G} = Aut(G)$, where Φ'_8 and Φ''_8 consist of vectors $\beta + m$ where m is a β-heptad.
If m is an α-heptad we have $\langle \alpha + m, x(m) \rangle = 0$ for the Higman-vector $x(m) = \alpha + m + B_1(m)$. Hence we conclude from (2.7) that we may choose the notation such that $\Phi'_8 \subseteq H'_{78}$ and $\Phi''_8 \subseteq H''_{78}$.
From (3.15) we have $W_8(H_{79}) = W_8(H_{78}) + 35.200$ and the assertion follows.However, it is easy to avoid the use of (3.15) in order to obtain the result.

We give a sketch of a short direct proof:
Let $y \in W_8(H_{79}) \setminus H_{78}$. Let $k(y) = \max\{\ |\Delta(\xi) \cap supp(y)|\ |\ \xi \in supp(y)\}$
Then $k(y) \in \{1,3,5,7\}$ by (3.16). The possibility $k(y) \in \{1,3,5\}$ is ruled out be easy contradictions. Therefore $k(y) = 7$ and it follows that $y = \beta + m$ for a subset $m \subseteq \Delta(\beta)$ of cardinality 7. Without loss we may assume $\beta = \alpha$ and we obtain from (1.3) that m is a heptad in ω_{22} from the fact that $\langle y, \Delta(\gamma) \rangle = 1$ for every $\gamma \in \Delta_2(\alpha)$. □

(3.18) LEMMA. Let $x \in H_{23} \setminus H_{22}$ and $\beta \in \Omega$.
Then $w(x\,\Delta(\beta)) = |supp(x) \cap \Delta(\beta)| \in \{7,11,15\}$.

Proof. The assertion follows from (3.6) and (1.3), since G acts transitively on Ω. □

(3.19) PROPOSITION. H'_{22} has minimum weight 32.

Proof. H_{21} is the subcode of H'_{22} consisting of all vectors of weight divisible by 4. Since 32 ist the minimum weight of H_{21} by (3.5) it suffices to show that $w_i(H'_{22}) = 0$ for all $0 < i < 32$.
Let $x \in W_i(H'_{22})$ where $0 < i \leq 32$. Let $\lambda_j = \lambda_j(x)$ for $\sigma \leq j \leq 22$.
Counting the edges of the Higman-Sims graph between supp (x) and Ω gives by (3.18) the equations

$$w(x)\cdot 22 = 7\lambda_7 + 11\lambda_{11} + 15\lambda_{15},$$
$$100 = \lambda_7 + \lambda_{11} + \lambda_{15}.$$

It follows $w(x)\cdot 22 = 700 + 4\lambda_{11} + 8\lambda_{15} \geq 700$ and $w(x) > 31$, hence $w(x) = 32$. □

We are now in a position to obtain the weight distributions of H'_{22} and H_{23} using the Mac-Williams or Pless identities.

(3.20) THEOREM. The weight distribution of H'_{22} and the orbits of G in H'_{22} are as described in the following table.

i	$w_i(H'_{22})$	length of G-orbits in $W_i(H'_{22})$
0/100	1	1
32/68	3.850	3.850
34/66	5.600	5.600
36/64	4.125	4.125
40/60	92.400	15.400, 77.000
42/58	387.200	17.600, 369.600
44/56	347.600	1.100, 346.500
48/52	600.600	231.000, 369.600
50	1.311.552	352, 123.200, 264.000, 924.000

Proof. (i) Let $a_i = w_i(H'_{22})$ and $b_i = w_i(H'_{78})$. We recall that $H'_{78} = (H'_{22})^{\perp}$. Hence the families (a_i) and (b_i) are related to each other by the MacWilliams identities.
We have $H'_{22} \leq H'_{78} \leq H_{99}$, as follows from (2.7); hence for every odd i $a_i = b_i = 0$. Since $\mathbb{1} \in H'_{22}$, we have $a_i = a_{100-i}$ for all i. Moreover $\{x \mid x \in H'_{22}$ and $w(x) \equiv 0 \pmod 4\} = H_{21}$.
In view of (3.19) we therefore have the following information:

$a_i = 0$ for $0 < i < 32$ and $78 < i < 100$;
$a_o = a_{100} = 1$ and from (3.5) every a_i with $i \equiv 0 \pmod 4$ is known.

So the only unknown weight numbers of H'_{22} with $i \leq 50$ are

$a_{34}, a_{38}, a_{42}, a_{46}$ and a_{50}.

On the other hand we know from (3.3) and (3.18) the values

$b_o = 1$
$b_2 = b_4 = b_6 = 0$ and
$b_8 = 17.600$.

It follows from the general theory of MacWilliams identities that the weight distribution (a_i) ist uniquely determined by the known values of the a_i and b_j, see e.g. [21, Theorem]. Explicit calculations yield the values asserted in the Theorem.

(ii) It remains to determine the G-orbits in $H'_{22} \setminus H_{21}$. This will be done in a sequence of lemmas, as some more detailed investigations are required. □

Recall from section 2 that $\overline{G} = \mathrm{Aut}(G)$ interchanges the codes H'_{22} and H''_{22}. Therefore we have a natural involutory correspondence between the G-orbits in $H'_{22} \setminus H_{21}$ and those in $H''_{22} \setminus H_{21}$. We agree that X' and X'' will always denote corresponding G-orbits in this sense.

We start by considering the Higman vectors $x(m) = \alpha + m + B_1(m)$ which we know to belong to $H_{23} \setminus H_{22}$ from (2.7). Recall that we denote the G_α-orbits $\mathfrak{M}'$ and $\mathfrak{M}''$ on the heptads of $\mathcal{W}_{22}$ such that $x(m) \in H'_{22}$ if and only if $m \in \mathfrak{M}'$.

(3.21) PROPOSITION. Let $X'_o = \{x(m) \mid m \in \mathfrak{M}'\} \cup \{x(m) + \mathbb{1} \mid m \in \mathfrak{M}'\}$. Then the following hold:

(1) X'_0 is a G-orbit in $W_{50}(H'_{22}) \setminus H_{21}$.
(2) $G_{x(m)} \cong PSU(3,5^2)$ acts as a rank 3 group on the supports of $x(m)$ and $x(m) + \mathbb{1}$; the supports of $x(m)$ and of $x(m) + \mathbb{1}$ are nonisomorphic $G_{x(m)}$ - spaces.
$G_{\{x(m),\, x(m)+\mathbb{1}\}} \cong P\Sigma U(3,5^2)$ acts transitively on Ω.
(3) $\mathrm{supp}\,(x(m)) = \Lambda_7(x(m))$ and $\mathrm{supp}\,(x(m) + \mathbb{1}) = \Lambda_{15}(x(m))$.
(4) The mapping $m \longmapsto \{x(m), x(m) + \mathbb{1}\}$ of $\mathfrak{M}'$ onto X'_0/H_1 is bijective and a G_α-morphism.
In particular $|X'_0| = 2 \cdot 176 = 352$.

Proof. It follows from (2.7) and (1.9) that $X'_0/H_1 = \{\{x, x+\mathbb{1}\} \mid w(x) = 50$ and the Higman-Sims graph induces on supp(x) and supp(x+𝟙) a strongly regular graph with valency 7$\}$. Therefore X'_0/H_1 and X'_0 are invariant under G. It is immediate that $m \mapsto \{x(m), x(m) + \mathbb{1}\}$ is a bijective morphism of G_α-spaces.

It follows that $|X_0'/H_1| = 176$ and that G acts transitively on X_0'. From (1.3) and (1.9) we infer that $G_{x(m),\beta} \cong \mathrm{Alt}(7)$ for all $\beta \in \Omega$ and that $G_{x(m),\beta}$ acts as a rank 3 group on the supports of $x(m)$ and $x(m) + \mathbb{1}$, the supports being nonisomorphic $G_{x(m)}$-spaces. From D.G. Higman's result [7.6.1] we may conclude that $G_{x(m)} = PSU(3,5^2)$ and that $G_{\{x(m),\, x(m)+\mathbb{1}\}} \cong P\Sigma U(3,5^2)$. The rest of the assertion easily follows. □

REMARK. Note that $P\Sigma U(3,5^2) \cong G_{\{x(m),\, x(m)+\mathbb{1}\}}$ does not leave invariant the strongly regular graphs induced on $x(m)$ and $x(m) + \mathbb{1}$, but interchanges them blockwise. It follows from (2.7) that $\overline{G}_{\{x(m),\, x(m)+\mathbb{1}\}} = G_{\{x(m),\, x(m)+\mathbb{1}\}}$ where $\overline{G} = \mathrm{Aut}(G)$; hence even $\overline{G}$ induces on the supports of $x(m)$ and of $x(m) + \mathbb{1}$ only the group $PSU(3,5^2)$, not the complete automorphism group $P\Sigma U(3,5^2)$ of the strongly regular graph of valency 7 on 50 vertices.

By the convention above there is a G-orbit X_0'' in $W_{50}(H_{22}'') \setminus H_{21}$ which shares analoguous properties. X_0' and X_0'' are interchanged by $\overline{G} = \mathrm{Aut}(G)$.

It is much more complicated to deal with the remaining G-orbits. We use the connections in the Higman-Sims-graph as a guide.

(3.22) LEMMA. Let $x = x(m) \in X_0'$; then $\alpha \in \Lambda_{15}(x+\mathbb{1})$. We have $y = (x+\mathbb{1}) + \Delta(\alpha) \in \Lambda_{42}(H_{22}'')$ and y has the following properties:

(1) $G_y = G_{x,\alpha} \cong \mathrm{Alt}(7)$.

(2) G_y has the following orbits in Ω:

$\Phi_0 = \{\alpha\} = \mathrm{supp}(x) \cap \Lambda_7(y)$, $|\Phi_0| = 1$,

$\Phi_1 = \mathrm{supp}(m) = \mathrm{supp}(x) \cap \Lambda_{15}(y)$, $|\Phi_1| = 7$,

$\Phi_2 = B_1(m) = \mathrm{supp}(x) \cap \Lambda_{11}(y)$, $|\Phi_2| = 42$,

$\Phi_3 = \Delta(\alpha)+m = \mathrm{supp}(y+\mathbb{1}) \cap \mathrm{supp}(x+\mathbb{1}) \subseteq \Lambda_7(y)$, $|\Phi_3| = 15$,

$\Phi_4 = B_3(m) = \mathrm{supp}(y) \cap \Lambda_7(y)$, $|\Phi_4| = 35$.

(3) The matrix of the Higman-Sims-graph with respect to $(\Phi_4)_{0\leq i\leq 4}$ is

$$\begin{bmatrix} 0 & 7 & 0 & 15 & 0 \\ 1 & 0 & 6 & 0 & 15 \\ 0 & 1 & 6 & 5 & 10 \\ 1 & 0 & 14 & 0 & 7 \\ 0 & 3 & 12 & 3 & 4 \end{bmatrix}$$

Proof. The assertion follows from (1.13) since G_y must leave invariant supp(y) and all $\Lambda_i(y)$. □

(3.23) PROPOSITION. Let $x = x(m) \in X_0'$ and $y = (x+\mathbb{1}) + \Delta(\alpha)$ and denote by Y_0'' the G-orbit containing y. Then $|Y_0''| = 17.600$.

Proof. It follows from (3.21) that $|Y_0''| = |G:G_y| = 352 \cdot 50 = 17.600$.

Note that every vector in Y_0'' (and Y_0') shares the properties of y described in (3.22). In particular, $\Lambda_{15}(y)$ is a unique G_y-orbit of length 7 for any $y \in Y_0''$. If $y \in Y_0''$ and $\beta \in \Lambda_{15}(y)$, then $z = y + \Delta(\beta) \in W_{34}(H_{22}')$ and G_z contains a subgroup isomorphic to Alt(6). We show that G acts transitively on $W_{34}(H_{22}')$ by arguments independent of the preceding discussion.

(3.24) LEMMA. Let $z \in W_{34}(H_{22}')$ and $\lambda_i = |\Lambda_i(z)|$. Then the following hold:

(1) $\lambda_{11} = 12$ and $\lambda_7 = 88$.

(2) G_z does not fix any point in Ω.

Proof. $\lambda_{15} = 0$ holds, since H_{22}' has minimum weight 32. The canonical equations obtained by counting edges

$$34.22 = 7\lambda_7 + 11\lambda_{11}$$
$$100 = \lambda_7 + \lambda_{11}$$

yield (1).

Assume that G_z fixes a point $\beta \in \Omega$. From the first part of Theorem (3.20) follows $5.600 \geq |G:G_z| = |G:G_\beta||G_\beta:G_z| = 100|G_\beta:G_z|$ and $|G_\beta:G_z| \leq 56$. Since $G_\beta \cong M_{22}$ either $G_\beta = G_z$ or $G_z = G_{\beta\gamma}$ for some point $\gamma \neq \beta$. By considering the action of G_β and $G_{\beta\gamma}$ on Ω we see that $|\Lambda_{11}(z)| = 12$ ist impossible, a contradiction against(1). □

(3.25) PROPOSITION. G acts transitively on $W_{34}(H'_{22}) = Z'$. Let $z \in Z'$. Then the following hold:

(1) $G_z \cong M_{11}$, the simple group of order 7.920.

(2) $\Lambda_{11}(z) \subseteq \mathrm{supp}(z)$ and G_z acts triply transitive on $\Lambda_{11}(z)$. If $\beta \in \Lambda_{11}(z)$, $G_{z,\beta} \cong PSL(2,11)$. The orbits of G_z are $\Phi_0 = \Lambda_{11}(z)$, $\Phi_1 = \mathrm{supp}(z) \cap \Lambda_7(z)$ and $\Phi_2 = \mathrm{supp}(\mathbb{1}+z)$ of lengths 12, 22 and 66.

(3) The matrix of the Higman-Sims graph with respect to $(\Phi_i)_{0 \le i \le 2}$ is

$$\begin{bmatrix} 0 & 11 & 11 \\ 6 & 1 & 15 \\ 2 & 5 & 15 \end{bmatrix} .$$

Proof. (i) From the first part of Theorem (3.20) we have $w_{34}(H'_{22}) = 5.600 \equiv 0 \pmod{11}$. Therefore there exists $z \in W_{34}(H'_{22})$ such that 11 divides $|G_z|$. From (2.4) we infer that any element of G_z of order 11 fixes exactly one point in $\Lambda_{11}(z)$ and acts fixed-point-freely on $\Lambda_7(z)$. Sinze G_z does not fix any point in Ω by (2.4), it follows that G_z acts doubly-transitively on $\Lambda_{11}(z)$.

(ii) Let $\beta \in \Lambda_{11}(z)$. Then $|G_z : G_{z,\beta}| = 12$ and we obtain that $|G_{z,\beta}| = |G|/(12\,|G:G_z|) \geq 44.352.000/(12 \cdot 5600) = 660$. On the other hand, it follows from (1.3) that $G_{z,\beta}$ is isomorphic to a subgroup of $PSL(2,11)$ whose order ist 660. Hence $G_{z,\beta} \cong PSL(2,11)$. Consequently G_z acts faithfully and triply transitively on $\Lambda_{11}(z)$ and $G_z \cong M_{11}$ by [24]. Another cosequence is $|G:G_z| = 5.600$ which implies that $Z' = W_{34}(H'_{22})$ is an orbit of G.

(iii) Since any element of G_z of order 11 fixes exactly one point in Ω, it follows that $\Lambda_{11}(z) \subseteq \mathrm{supp}(z)$. From the properties of the Higman-Sims graph we infer that G_z acts transitively on $\mathrm{supp}(z) \cap \Lambda_{11}(z)$. Without loss we may assume that $\alpha \in \Lambda_{11}(z)$. Using the notation of (1.12) we then may conclude that for the endecad $e = \Delta(\alpha) \cap x$ we have $B_3(e) \subseteq \mathrm{supp}(z+\mathbb{1})$. Since $G_{z,\alpha}$ acts transitively on $B_3(e)$, it now easily follows that G_z acts transitively on $\mathrm{supp}(z+\mathbb{1})$ of cardinality 66. The graph matrix in the assertion is now obtained by simple counting. □

REMARK. A vector $z \in Z'$ may be constructed explicitly as follows: Let e be an endecad such that e and any heptad $m \in \mathfrak{M}'$ generate the same M_{22}-invariant subcode of the (shortened) Golay-code of length 22. Then $z = \alpha + e + B_1(e) + B_5(e) \in Z'$ and $z+\mathbb{1} = (b + \Delta(\alpha)) + B_3(e)$, see (1.12).
It is now quite obvious how the orbits Z' and Y''_0 are linked by the Higman-Sims graph:

$$Y''_0 = \{z + \Delta(\beta) \mid z \in Z' \text{ and } \beta \in \mathrm{supp}(z) \cap \Lambda_7(z)\} \text{ and}$$
$$Z' = \{y + \Delta(\gamma) \mid y \in Y''_0 \text{ and } \gamma \in \Lambda_{15}(y)\}.$$

It is clear that we may construct a second orbit Y''_1 in $W_{42}(H''_{22})$ starting from Z' by making use of the G_z-orbit $\mathrm{supp}(z+\mathbb{1})$ instead of $\mathrm{supp}(z) \cap \Lambda_7(z)$.

(3.26) PROPOSITION. Let $z \in Z' = W_{34}(H'_{22})$ and let $\gamma \in \mathrm{supp}(z+\mathbb{1})$. Then $y = z + \Delta(\gamma) \in W_{42}(H''_{22})$. Denote by Y''_1 the G-orbit containing y.
The following assertions hold.

(1) $G_y = G_{z,\gamma} \cong \Sigma_5$.

(2) $|Y''_1| = 369.600$.

(3) G_y has exactly 9 orbits $(\Psi_i)_{0 \le i \le 8}$ in Ω which are defined as follows:
$\Psi_0 = \Lambda_{11}(z) \cap \Delta(\gamma)$; $|\Psi_0| = 2$ and $\Psi_0 \subseteq \Lambda_{11}(y)$.
$\Psi_1 = \Lambda_{11}(z) \setminus \Psi_0$; $|\Psi_1| = 10$ and $\Psi_1 \subseteq \Lambda_{11}(y)$.
$\Psi_2 = (\Lambda_7(z) \cap \mathrm{supp}(z)) \cap \Delta(\gamma)$; $|\Psi_2| = 5$ and $\Psi_2 \subseteq \Lambda_7(y)$.
Ψ_3, $\Psi_4 \subseteq \Lambda_7(z) \cap \mathrm{supp}(z) \setminus \Delta(\gamma)$ such that
$|\Psi_3| = 5$ and $|\Psi_4| = 12$; $\Psi_3 \subseteq \Lambda_7(y)$ and $\Psi_4 \subseteq \Lambda_{11}(y)$.
$\Psi_5 = \{\gamma\} = \Lambda_{15}(y)$.
$\Psi_6 = \mathrm{supp}(z+\mathbb{1}) \cap \Delta(\alpha)$; $|\Psi_6| = 15$ and $\Psi_6 \subseteq \Lambda_7(y)$.
Ψ_7, $\Psi_8 \subseteq \mathrm{supp}(z+1) \setminus \Delta(\gamma)$ such that
$|\Psi_7| = 20$ and $|\Psi_8| = 30$; $\Psi_7 \subseteq \Lambda_7(y)$ and $\Psi_8 \subseteq \Lambda_{11}(y)$.

(4) $\mathrm{Supp}\,(y) = \Psi_1 \cup \Psi_3 \cup \Psi_4 \cup \Psi_6.$
$\Lambda_7\,(y) = \Psi_2 \cup \Psi_3 \cup \Psi_6 \cup \Psi_7.$

(5) The matrix of the Higman-Sims graph with respect to $(\Psi_i)_{0 \leq i \leq 8}$ is

$$\begin{bmatrix} 0 & 0 & 0 & 5 & 6 & 1 & 0 & 10 & 0 \\ 0 & 0 & 3 & 2 & 6 & 0 & 3 & 2 & 6 \\ 0 & 6 & 0 & 1 & 0 & 1 & 0 & 8 & 6 \\ 2 & 4 & 1 & 0 & 0 & 0 & 3 & 0 & 12 \\ 1 & 5 & 0 & 0 & 1 & 0 & 5 & 5 & 5 \\ 2 & 0 & 5 & 0 & 0 & 0 & 15 & 0 & 0 \\ 0 & 2 & 0 & 1 & 4 & 1 & 0 & 4 & 10 \\ 1 & 1 & 2 & 0 & 3 & 0 & 3 & 3 & 9 \\ 0 & 2 & 1 & 2 & 2 & 0 & 5 & 6 & 4 \end{bmatrix}.$$

Proof. It follows from (3.25) and [3,Table 3] that $G_{z,\gamma} \cong \Sigma_5$ and, using the character table of M_{11}, we obtain that $G_{z,\gamma}$ has the nine orbits $\Psi_i (0 \leq i \leq 8)$ in Ω as described in assertion (3). From (3.25) we derive the graph matrix giben in (5) and it follows that $\Lambda_{15}(y) = \{ \} = \Psi_5$. Therefore $G_y = G_{z,\gamma}$ and the rest of the assertion easily follows. □

Since $W_{42}(H''_{22}) = 17.600 + 369.600$ by the first part of Theorem (3.20) only the remaining orbits in $W_{50}(H'_{22})$ have to be determined. We use for the construction of these orbits the same ideas as for the construction of Y_1''. We prove first a general lemma.

(3.27) LEMMA. Let $x \in W_{50}(H'_{22})$ and $\lambda_i = |\Lambda_i(x)|$.
Then $\lambda_7 = \lambda_{15}$.

Proof. Counting the edges of the Higman-Sims graph between supp(x) and Ω yields the equations

$$1.100 = 50 \cdot 22 = 7\lambda_7 + 11\lambda_{11} + 15\lambda_{15}$$
$$100 = \lambda_7 + \lambda_{11} + \lambda_{15}.$$

It follows $\lambda_7 = \lambda_{15}$. □

We consider the incidence structure

$$\mathfrak{J} = (W_{42}(H''_{22}),\ W_{50}(H'_{22}),\ I)$$

where $I = \{ (y,x) \mid y \in W_{42}(H''_{22}),\ x \in W_{50}(H'_{22})$ and there exists a $\beta \in \Omega$ such that $x + y = \Delta(\beta) \}$.

Note that $(y,x) \in I$ implies that $x + y = \Delta(\beta)$ where $\beta \in \Lambda_7(y) \cap \Lambda_{15}(x)$. It is obvious from the definition of $\mathfrak{J}$ that G acts on $\mathfrak{J}$ as a group of automorphisms. We know that G has exactly 2 orbits Y''_o and Y''_1 in $W_{42}(H''_{22})$ of lengths 17.600 and 369.600 respectively. It is our goal to determine the G-orbits in $W_{50}(H'_{22})$ via the G-orbits in I.

(3.28) PROPOSITION. G has precisely 7 orbits in I:

$I_{oo} = \{(y, y + \Delta(\beta)) \mid y \in Y''_o$ and $\beta \in \Phi_o(y)\}$,
$I_{o1} = \{(y, y + \Delta(\beta)) \mid y \in Y''_o$ and $\beta \in \Phi_3(y)\}$,
$I_{o2} = \{(y, y + \Delta(\beta)) \mid y \in Y''_o$ and $\beta \in \Phi_4(y)\}$,
$I_{10} = \{(y, y + \Delta(\beta)) \mid y \in Y''_1$ and $\beta \in \Psi_2(y)\}$,
$I_{11} = \{(y, y + \Delta(\beta)) \mid y \in Y''_1$ and $\beta \in \Psi_3(y)\}$,
$I_{12} = \{(y, y + \Delta(\beta)) \mid y \in Y''_1$ and $\beta \in \Psi_6(y)\}$,
$I_{13} = \{(y, y + \Delta(\beta)) \mid y \in Y''_1$ and $\beta \in \Psi_7(y)\}$.

(Here we write $\Phi_i(y) = \Phi_i$ in the sense of (3.22) and $\Psi_i(y) = \Psi_i$ in the sense of (3.26) for the sake of clarity.)

The orbit lengths are

$|I_{oo}| = 17.600 \cdot 1 = 17.600$
$|I_{o1}| = 17.600 \cdot 15 = 264.000$
$|I_{o2}| = 17.600 \cdot 35 = 616.000$
$|I_{10}| = 369.600 \cdot 5 = 1.848.000$
$|I_{11}| = 369.600 \cdot 5 = 1.848.000$
$|I_{12}| = 369.600 \cdot 15 = 5.544.000$
$|I_{13}| = 369.600 \cdot 20 = 7.392.000$

Proof. The assertion is a straighforward consequence of (3.22) and (3.26). Note that for $y \in W_{42}(H''_{22})$ we have $y + \Delta(\beta) \in W_{50}(H'_{22})$ if and suly if $\beta \in \Lambda_7(y)$. □

An immediate consequence of (3.28) is that there are at most 7 G-orbits in $W_{50}(H'_{22})$. We know one of these G-orbits, X'_o, from (3.21) X'_o corresponds uniquely to I_{oo}, as follows from (3.21). From the first part of Theorem (3.20) we know that $W_{50}(H'_{22})$ = 1.311.552. We construct now the remaining G-orbits in $W_{50}(H'_{22})$ by considering some particular vectors.

(3.29) PROPOSITION. Let $y = (x(m)+\mathbb{1}) + \Delta(\alpha) \in Y''_o$ as defined in (3.22), let $\beta \in \Phi_3(y) = \Phi_3$ and let $x = y + \Delta(\beta)$. Set $X'_1 = \{xg \mid g \in G\}$. Then the following hold:

(1) $G_x = G_{y,\beta} \cong PSL(2,7)$

(2) $|X'_1| = 264.000$.

(3) G_x has precisely 8 orbits $\Theta_i\,(0 \leq i \leq 7)$ in Ω:

$\Theta_o = \{\alpha\}$, $|\Theta_o| = 1$ and $\Theta_o \subseteq \Lambda_7(x)$;

$\Theta_1 = supp(m)$, $|\Theta_1| = 7$ and $\Theta_1 \subseteq \Lambda_{15}(x)$;

$\Theta_2 = supp(y) \cap \Delta(\beta) \setminus \{\alpha\}$, $|\Theta_2| = 14$ and $\Theta_2 \subseteq \Lambda_{11}(x)$;

$\Theta_3 = B_1(m) \setminus \Delta(\beta)$, $|\Theta_3| = 28$ and $\Theta_3 \subseteq \Lambda_{11}(x)$;

$\Theta_4 = \{\beta\}$, $|\Theta_4| = 1$ and $\Theta_4 \subseteq \Lambda_{15}(x)$;

$\Theta_5 = \Delta(\alpha) \setminus supp(m) \setminus \{\beta\}$, $|\Theta_5| = 14$ and $\Theta_5 \subseteq \Lambda_{11}(x)$;

$\Theta_6 = \Delta(\beta) \cap supp(y)$, $|\Theta_6| = 7$ and $\Theta_6 \subseteq \Lambda_7(x)$;

$\Theta_7 = B_3(m) \setminus \Delta(\beta)$, $|\Theta_7| = 28$ and $\Theta_7 \subseteq \Lambda_{11}(x)$.

$supp(x) = \Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_7$.

(4) The matrix of the Higman-Sims graph with respect to $(\Theta_i)_{0 \leq i \leq 7}$ is

$$\begin{bmatrix} 0 & 7 & 0 & 0 & 1 & 14 & 0 & 0 \\ 1 & 0 & 2 & 4 & 0 & 0 & 3 & 12 \\ 0 & 1 & 0 & 6 & 1 & 4 & 0 & 10 \\ 0 & 1 & 3 & 3 & 0 & 5 & 3 & 7 \\ 1 & 0 & 14 & 0 & 0 & 0 & 7 & 0 \\ 1 & 0 & 4 & 10 & 0 & 0 & 1 & 6 \\ 0 & 3 & 0 & 12 & 1 & 2 & 0 & 4 \\ 0 & 3 & 5 & 7 & 0 & 3 & 1 & 3 \end{bmatrix}.$$

Proof. We make use of the results of (3.22): $G_y \cong Alt(7)$ and $|G_y : G_{y,\beta}| = 15$, hence $G_{y,\beta} \cong PSL(2,7)$. Using the information given in (3.22) we easily obtain that $G_{y,\beta}$ has the orbits $\Theta_i\,(0 \leq i \leq 7)$, in Ω. Counting the edges of the Higman-Sims graph yields the graph matrix in (4) and shows that every Θ_i is left invariant by G_x. It follows that $G_x = G_{y,\beta}$, and hence $|X'_1| = 264.000$. The rest of the assertion is now obvious. □

REMARK. It can be deduced from (3.21) and (3.22) that x and $x + \mathbb{1}$ are in the same G-orbit X'_1. This will follow also by simple numerical reasons when the proof of Theorem (3.20) is complete. It should be noted that $G_{\{x,x+\mathbb{1}\}} \cong PGL(2,7)$ and that the Higman-Sims graph induces on $\Theta_1 \cup \Theta_6$ the incidence graph of the projective plane of order 2, displaying in this way the well known isomorphy $PSL(2,7) \cong PSL(3,2)$ and $PGL(2,7)$ as correlation group of the projective plane of order 2.

(3.30) LEMMA. Let $y = (x(m)+\mathbb{1}) + \Delta(\alpha) \in Y''_o$ as defined in (3.22), let $\gamma \in \Phi_4(y) = \Phi_4$ and let $x = y + \Delta(\gamma)$. Set $X'_2 = \{xg \mid g \in G\}$. Let $a = |G_x : G_{y,\gamma}|$.
Then the following hold:

(1) $G_{y,\gamma} \cong \Sigma_4 \rtimes \Sigma_3$.

(2) $|X'_2| = 616.000/a$.

(3) $G_{y,\gamma}$ has precisely 12 orbits Ξ_i $(0 \leq i \leq 11)$ in Ω:

$\Xi_0 = \{\alpha\}$, $|\Xi_0| = 1$ and $\Xi_0 \subseteq \Lambda_7(x)$;

Ξ_1, Ξ_2 are the orbits of $G_{y,\gamma}$ in m such that

$|\Xi_1| = 4$ and $|\Xi_2| = 3$; $\Xi_1 \cup \Xi_2 \subseteq \Lambda_{15}(x)$;

$\Xi_3 = \Delta(\gamma) \cap B_1(m)$, $|\Xi_3| = 12$ and $\Xi_3 \subseteq \Lambda_{11}(x)$;

Ξ_4, Ξ_5 are the orbits of $G_{y,\gamma}$ in $B_1(m) \setminus \Delta(\gamma)$ such that

$|\Xi_4| = 12$ and $|\Xi_5| = 18$.

Ξ_6, Ξ_7 are the orbits of $G_{y,\gamma}$ in $\Delta(\alpha) \setminus m$ such that

$|\Xi_6| = 12$ and $|\Xi_7| = 3$.

$\Xi_8 = \{\gamma\}$, $\Xi_8 = 1$ and $\Xi_8 \subseteq \Lambda_{15}(x)$.

$\Xi_9 = \Delta(\gamma) \cap B_3(m)$, $|\Xi_9| = 4$ and $\Xi_9 \subseteq \Lambda_7(x)$.

Ξ_{10}, Ξ_{11} are the orbits of $G_{y,\gamma}$ in $B_3(m) \setminus (\{\gamma\} \cup \Delta(\gamma))$

such that $|\Xi_{10}| = 12$ and $|\Xi_{11}| = 18$.

$$\mathrm{supp}(x) = \Xi_1 \cup \Xi_3 \cup \Xi_7 \cup \Xi_8 \cup \Xi_{10} \cup \Xi_{11}.$$

(4) The matrix of the Higman-Sims graph with respect to $(\Xi_i)_{0 \leq i \leq 11}$ is

0	4	3	0	0	0	12	3	0	0	0	0
1	0	0	3	3	0	0	0	0	3	3	9
1	0	0	0	0	6	0	0	1	0	8	6
0	1	0	0	3	3	5	0	1	0	3	6
0	1	0	3	0	3	3	2	0	1	3	6
0	0	1	2	2	2	4	1	0	2	4	4
1	0	0	5	3	6	0	0	0	1	3	3
1	0	0	0	8	6	0	0	1	0	0	6
0	0	3	12	0	0	0	3	0	4	0	0
0	3	0	0	3	9	3	0	1	0	3	0
0	1	2	3	3	6	3	0	0	1	0	3
0	2	1	4	4	4	1	0	0	0	2	2

(5) $\Lambda_{15}(x) \cap \mathrm{supp}(x) = \Xi_1 \cup \Xi_8$ has 5 points,

$\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1}) = \Xi_2 \cup \Xi_4$ has 15 points.

G_x has at least 2 orbits in $\Lambda_{15}(x)$.

(6) $|\Lambda_7(x)| = |\Lambda_{15}(x)| = 20$,

$|\Lambda_{11}(x)| = 60$.

<u>Proof.</u> ((1) follows from (3.22), since $\gamma \in \Phi_4(y)$. (2) follows from (1). The orbits Ξ_i and the graph matrix are obtained by studying the action of $G_{y,\gamma} \cong \Sigma_4 \curlywedge \Sigma_3$ using (3.22). The remaining part of the assertion follows by inspection. □

(3.31) LEMMA. Let $y \in Y_1''$ as defined in (3.26) and let $\delta \in \Psi_6(y) = \Psi_6$. Set $x = y + \Delta(\delta)$ and $X_3' = \{ xg \mid g \in G\}$. Let $b = |G_x : G_{y,\delta}|$.
Then the following hold:

(1) $G_{y,\delta} \cong D_8$.

(2) $|X_3'| = 5.544.000/b$.

(3) $G_{y,\delta}$ has precisely 26 orbits on Ω, 13 in supp(x) and 13 in supp(x+$\mathbb{1}$).

(4) $x + \mathbb{1} \in X_3'$.

(5) $|\Lambda_{15}(x) \cap \mathrm{supp}(x)| = 6$ and $|\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1})| = 8$.

$|\Lambda_7(x)| = |\Lambda_{15}(x)| = 14$ and

$|\Lambda_{11}(x)| = 72$.

G_x has at least 2 orbits in $\Lambda_{15}(x)$.

<u>Proof.</u> It follows from (3.26) that $G_{y,\delta} \cong D_8$; hence (1) holds and also (2) because of $G_{y,\delta} \leq G_x$. From (3.26) we also obtain that $G_{y,\delta}$ has 13 orbits (of lengths 1, 1, 2, 2, 4, 4, 4, 4, 4, 4, 4, 8, 8) in supp(x) and 13 orbits (with the same lengths) in supp(x+$\mathbb{1}$). One easily checks that $\Lambda_7(x) \cap \mathrm{supp}(x) \neq \emptyset$ and $\Lambda_7(x) \cap \mathrm{supp}(x+\mathbb{1}) \neq \emptyset$. Therefore G_x has at least 2 orbits in $\Lambda_7(x)$. It now follows from (3.28) that G has exactly the orbits

X'_0, X'_1, X'_2 and X'_3 in $W_{50}(H'_{22})$ and that every $x \in W_{50}(H'_{22})$ is in the same G-orbit as its complement $x+\mathbb{1}$. One checks with the aid of (3.26) that $\Lambda_7(x) \cap \mathrm{supp}(x)$ contains 2 $G_{Y,\delta}$-orbits of length 4 each and that $\Lambda_{15}(x) \cap \mathrm{supp}(x)$ contains 3 $G_{Y,\delta}$-orbits of length 1, 1, 4; all remaining $G_{Y,\delta}$-orbits in supp(x) are in $\Lambda_{11}(x)$. Since $x+\mathbb{1} \in X'_3$ we obtain (5). □

REMARK. For the proof of (3.31) it is not necessary to compute completely the matrix of the Higman-Sims graph with respect to $G_{Y,\delta}$. Indirect arguments are sufficient. (Note that an explicit computation of a vector $x \in X'_3$ easily verifies the assertions.)
As another result of the proof of (3.31) we note the following:

(3.32) PROPOSITION. The G-orbits in $W_{50}(H'_{22})$ are X'_0, X'_1, X'_2, X'_3. For all $x \in W_{50}(H'_{22})$ the vectors x and $x+\mathbb{1}$ are in the same G-orbit. □

It remains to determine the lengths of the orbits X'_2 and X'_3 and the structure of the stabilizers.

(3.33) PROPOSITION. $|X'_3| = 924.000$ and $G_x \cong \Sigma_4 \times Z_2$ for $x \in X'_3$. G_x has exactly 2 orbits in $\Lambda_{15}(x)$, namely $\Lambda_{15}(x) \cap \mathrm{supp}(x)$ of length 6 and $\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 8.

Proof. Let $x \in X'_3$ as defined in (3.31). It follows from (3.28) and the known properties of Y''_i and X'_j that G_x has the orbits $\Lambda_{15}(x) \cap \mathrm{supp}(x)$ of length 6 and $\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 8 in $\Lambda_{15}(x)$. It follows that $b = |G_x : G_{Y,\delta}| = 6$, therefore $|X'_3| = 924.000$ and $|G_x| = 48$. (Note that $\delta \in \Lambda_{15}(x) \cap \mathrm{supp}(x)$.) That $G_x \cong \Sigma_4 \times Z_2$ is obtained by considering the action on $\Lambda_{15}(x) \cap \mathrm{supp}(x)$: G_x is isomorphic to a transitive subgroup of Σ_6 having index 15; all subgroups of Σ_5 of index 15 are conjugate in $\mathrm{Aut}(\Sigma_6) \cong P\Gamma L(2,9)$ (not in Σ_6!), therefore $G_x \cong \Sigma_4 \times \Sigma_2$ the stabilizer in Σ_6 of a 2-element set. □

(3.34) PROPOSITION. $|X'_2| = 123.200$ and $G_x \cong \Sigma_5 \curlywedge^2 \Sigma_3$ for $x \in X'_2$. G_x has exactly 2 orbits in $\Lambda_{15}(x)$, namely $\Lambda_{15}(x) \cap \mathrm{supp}(x)$ of length 5 and $\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 15.

Proof. We may assume that $x \in X'_2$ is as defined in (3.30). The assertion is now easily obtained using (3.28), (3.30) and (3.33). □

REMARK. It is not hard to show that in (3.34) the stabilizer G_x has the following orbits

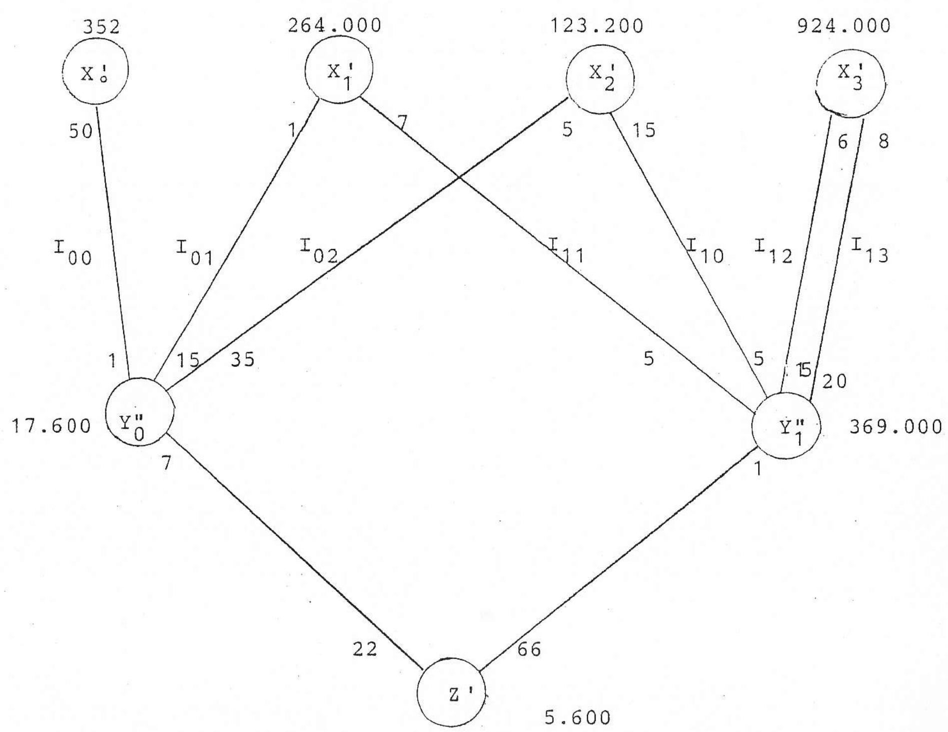
$\Lambda_{15}(x) \cap \mathrm{supp}(x)$ of length 5,
$\Lambda_{7}(x) \cap \mathrm{supp}(x)$ of length 15,
$\Lambda_{11}(x) \cap \mathrm{supp}(x)$ of length 30,
$\Lambda_{7}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 5,
$\Lambda_{15}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 15 and
$\Lambda_{11}(x) \cap \mathrm{supp}(x+\mathbb{1})$ of length 30.

The matrix of the Higman-Sims graph with respect to these G_x-orbits (in that order) is

$$\begin{bmatrix} 0 & 3 & 12 & 4 & 3 & 0 \\ 1 & 0 & 6 & 1 & 8 & 6 \\ 2 & 3 & 6 & 0 & 3 & 8 \\ 4 & 3 & 0 & 0 & 3 & 12 \\ 1 & 8 & 6 & 1 & 0 & 6 \\ 0 & 3 & 8 & 2 & 3 & 6 \end{bmatrix}$$

One may check that this result agrees with Lemma (3.30).

The proof of Theorem (3.20) is now complete. We conclude this section by a diagramm which displays the G-invariant relations between the orbits Z', Y''_i and X'_j given by addition of elements $\Delta(\xi)$, $\xi \in \Omega$.



The numbers at the ends of the strokes indicate the length of the stabilizer-orbit belonging to the relation orbit of G. Note that Y''_1 is joined to X'_2 via the stabilizer orbit Ψ_2 (see (3.26) and (3.28)).

(3.25) COROLLARY. The weigth distribution of H_{23} is as described in the following tabel:

i	$w_i(H_{23})$
0/100	1
22/78	100
30/70	1.100
32/68	3.850
34/66	11.200
36/64	4.125
38/62	38.500
40/60	92.400
42/58	968.000
44/56	347.600
46/54	485.100
48/52	600.600
50	3.283.456

Proof. The assertion follows from the known structure of H_{23}/H_{21} together with (3.11) and (3.20). □

The weight enumerators of H_{77}, H'_{78} and H''_{78} can finally be determined via MacWilliams transformation. (The computations have been carried out at the Rechenzentrum der Universität Tübingen by F.H. Florian using an ALDES program.)

(3.36) PROPOSITION. The weight distributions of H_{77}, H'_{78} and H''_{78} are as described in the following tables.

i	$w_i(H_{77})$
0/100	1
8/92	119 625
10/90	3 351 040
12/88	262 194 275
14/86	10 460 595 200
16/84	321 165 892 575
18/82	7 309 692 544 000
20/80	127 793 807 058 995
22/78	1 748 088 230 732 800
24/76	19 020 111 451 577 775
26/74	166 791 619 843 340 800
28/72	1 191 810 146 845 445 325
30/70	7 002 911 342 735 052 800
32/68	34 096 837 242 289 671 850
34/66	138 453 825 199 499 980 800
36/64	471 402 307 704 520 229 125
38/62	1 351 845 015 778 272 153 600
40/60	3 277 357 630 135 681 557 850
42/58	6 737 425 031 983 982 617 600
44/56	11 772 688 854 024 011 418 750
46/54	17 516 850 935 961 851 443 200
48/52	22 222 175 416 214 614 898 750
50	24 054 370 909 850 203 084 800

i	$w_i(H'_{78}) = w_i(H''_{78})$
0/100	1
8/92	137 225
10/90	7 231 840
12/88	511 741 475
14/86	20 997 046 400
16/84	642 104 180 575
18/82	14 620 696 059 200
20/80	255 580 729 090 995
22/78	3 496 191 224 323 200
24/76	38 040 223 036 760 175
26/74	333 583 215 539 666 400
28/72	2 383 620 193 904 285 325
30/70	14 005 823 013 894 187 520
32/68	68 193 674 589 734 420 650
34/66	276 907 649 362 395 385 600
36/64	942 804 613 467 381 809 925
38/62	2 703 690 046 024 936 460 800
40/60	6 554 715 226 694 274 576 090
42/58	13 474 850 114 631 510 264 000
44/56	23 545 377 636 355 278 743 550
46/54	35 033 701 987 289 528 723 200
48/52	44 444 350 658 167 367 673 150
50	48 108 742 023 087 188 141 952

REMARK. From (2.7) it directly follows that there exist G-invariant linear forms f', f" of H_{79} such that $xf' \neq 0 \neq xf''$ for all $x \in H_{78} \setminus H_{77}$ and all $x \in H_{22} \setminus H_{21}$. The results of section 3 therefore imply that we may obtain by adding two "parity checks" a (binary) (102,79)-code of minimum weight 8 and a (102,23)-code of minimum weight 24.

It should also be pointed out that by (3.35) about 38% of the elements of H_{23} are vectors of weight 50 which may be considered as a kind of "pseudo-noise sequences".

4. A model of G.Higman's geometry

In this section we consider the embedding of the Higman-vectors $x(m) = \alpha + m + B_1(m)$ in the code H_{23} and derive in this way a natural model of G.Higman's geometry [10] on which the Higman-Sims group acts as a group of automorphisms. We thereby obtain an easy direct proof that G.Higman's simple group [10] is in fact isomorphic to the Higman-Sims group. Older proofs of this well known fact involve computer calculations [23], the use of the Leech lattice [3] or rather complicated combinatorial investigations [25,26] (For another recent elementary proof of the isomorphy see [2].) The code theoretic construction of G.Higman's geometry also provides for a simple explanation of G.Higmans "natural correspondence" between the unordered pairs of points and quadrics, not induced by a bijection.

We shall consider all possible sums $x(m_1) + x(m_2)$ of Higman-vectors. Recall that the M_{22}-orbits on the set of heptads of $\mathcal{W}_{22}$ are denoted by $\mathfrak{M}'$ and $\mathfrak{M}''$ and the notation for the codes is chosen so that $x(m) \in H'_{22}$ if and only if $m \in \mathfrak{M}'$. The additive structure of H_{23}/H_2 gives the following fact.

(4.1) LEMMA. Let $m_1, m_2 \in \mathfrak{M}' \cup \mathfrak{M}''$.

(1) $x(m_1) + x(m_2) \in H_{21}$ if and only if $|\{m_1, m_2\} \cap \mathfrak{M}'|$ is even.

(2) $x(m_1) + x(m_2) \in H_{22} \setminus H_{21}$ if and only if $|\{m_1, m_2\} \cap \mathfrak{M}'| = 1$. [

More precise information is given by computing the weights:

(4.2) LEMMA. Let $m_1, m_2 \in \mathfrak{M}' \cup \mathfrak{M}''$ and let

$d = w(m_1 m_2) = |\mathrm{supp}(m_1) \cap \mathrm{supp}(m_2)|$.

Then $w = w(x(m_1) + x(m_2))$ is given by the following table:

d	0	1	2	3	4	7
w	70	60	50	40	30	0

Proof: The assertion easily follows from the definition of $x(m_i)$ by using the Leech triangle [3, p.226]. (Note that the heptads may be considered as shortened octads of the (extended) Golay code of length 24.) □

The results in (4.2) become more symmetrical when we pass to the factor space H_{23}/H_1 of complementary vectors. For convenience of notation let $\hat{x} = \{x, x + \mathbb{1}\} \in H_{23}/H_1$ for $x \in H_{23}$ and let $\hat{x}(m) = \widehat{x(m)}$. As before in section 3 the weight $w(\hat{x})$ of $\hat{x}$ is defined by $w(\hat{x}) = \{w(x), w(x + \mathbb{1})\}$.

(4.3) COROLLARY. The weight $\hat{w} = w(\hat{x}(m_1) + \hat{x}(m_2))$ as a function of $d = w(m_1 m_2)$ is given by the following table:

d	0	4	2	1	3	7
w	$\{30,70\}$	$\{30,70\}$	$\{50,50\}$	$\{40,60\}$	$\{40,60\}$	$\{0,100\}$

□

In (4.3) the table is ordered according to (4.1). We see that the cases d = 0 and d = 4 (resp. d = 1 and d = 3) yield the same weights. In the following we shall use the G-orbit structure known from section 3 to explain this observation.

From section 3 we know that G has exactly one orbit X in H_{23}/H_1 of elements of weight $\{30,70\}$ of length $|X| = 1.100$ and that G has exactly 2 orbits in H_{23}/H_1 of elements of weight $\{40,60\}$, one of them, say Y, of length $|Y| = 15.400$, the other of length 77.000. Moreover, G has exactly 2 orbits in H_{22}/H_1 of elements of weight $\{50,50\}$, one of them, say Z, of length $|Z| = 22.176$, the other of length 308.000. All these orbits are also $\bar{G}$-invariant.

In addition we set $X' = X_o'/H_1$ and $X'' = X_o''/H_1$ where X_o' and X_o'' are the G-orbits of Higman-vectors such that $X_o' \subseteq H'_{22}$ and $X_o'' \subseteq H''_{22}$. We have $|X'| = |X''| = 176$. Note that G acts transitively on X' and X'' and that the stabilizer in G of an element of $X' \cup X''$ is isomorphic to $P\Sigma U(3,5^2)$. $\bar{G} = Aut(G)$ acts transitively on $X' \cup X''$.

(4.4) LEMMA. Let $x_1, x_2 \in X' \cup X''$.
Then $x_1 + x_2 \in X \cup Y \cup Z \cup \{0\}$.

Proof: In view of corollary (4.3) we have to show only that $x_1 + x_2$ does not belong to the orbit of length 77.000 of elements of weight $\{40,60\}$ and not to the orbit of length 308.000 of elements of weight $\{50,50\}$. But this claim follows from (1.3) and (4.3), as $176(105 + 70) < 77.000$ and $176 \cdot 126 < 308.000$. □

As a consequence of (4.4) we may study the ternary relation $R = \{(x_1,x_2,x_1+x_2) \mid x_1,x_2 \in X' \cup X''\} \subseteq (X' \cup X'')^2 \times (X \cup Y \cup Z \cup \{0\})$ in some detail. Of course, this relation is $\bar{G}$-invariant.

(4.5) PROPOSITION.

(1) $R' = \{(x_1,x_2,x_1+x_2) \mid x_1,x_2 \in X' \text{ and } x_1 \neq x_2\}$ is a G-orbit in $(X')^2 \times Y$ of length $176 \cdot 175 = 30.800 = 2 \cdot 15.400$.

(2) $R'' = \{(x_1,x_2,x_1+x_2) \mid x_1,x_2 \in X'' \text{ and } x_1 \neq x_2\}$ is a G-orbit in $(X'')^2 \times Y$ of length $176 \cdot 175 = 30.800 = 2 \cdot 15.400$.

(3) R' and R'' are interchanged by $\bar{G}$.

Proof: The assertion follows from (4.4), (3.11) and (3.20). □

(4.6) COROLLARY. G acts doubly-transitively on X' and X''.

Proof: G acts transitively on Y. From (4.5) it follows that G is 2-homogeneons on X' and X''. Since |G| is even, the double-transitivity of G follows. □

(4.7) COROLLARY. There is a G-invariant natural correspondence

$$\Theta : X'^{\{2\}} \longrightarrow X''^{\{2\}} \text{ given by}$$

$$\{x_1,x_2\}\Theta = \{y_1,y_2\} \text{ where } y_1 + y_2 = x_1 + x_2.$$

Proof: The assertion also follows from (4.5). □

Note that Θ is not induced by a bijection $X' \to X''$, since X' and X'' are nonisomorphic G-sets.

Another particular G-orbit in R can be used to construct a model of G. Higman's geometry.

(4.8) PROPOSITION

(1) $R_1 = \{(x_1,x_2,x) \mid x_1 \in X', x_2 \in X'', x \in X \text{ and } x = x_1+x_2\}$

is a G-orbit in $X' \times X'' \times X$ of length

$$176 \cdot 50 = 8.800 = 8 \cdot 1.100.$$

(2) $R_2 = \{(x_1,x_2,x) \mid x_1 \in X'', x_2 \in X', x \in X \text{ and } x = x_1+x_2\}$

is a G-orbit in $X'' \times X' \times X$ of length

$$176 \cdot 50 = 8.800 = 8 \cdot 1.100.$$

(3) R_1 and R_2 are interchanged by $\bar{G}$.

Proof: The assertion follows from (4.4) and section 3. □

(4.9) COROLLARY. The stabilizer in G of an element $x \in X$ is isomorphic to the symmetric group Σ_8.

Proof: $|X| = 1.100$ implies $|G_x| = 40.320 = 8! = |\Sigma_8|$. The assertion now follows from (4.8). □

We are now in a position to define a model of G. Higman's geometry: Call the elements of X' points, the elements of X conics and the elements of X" quadrics.

The G-invariant relation R_1 (or equivalenty R_2) induces the following incidence structures by coordinate restriction:

$\mathfrak{F}' = (X',X,I_{ind})$ (point-conic-structure) ,

$\mathfrak{F}'' = (X'',X,I_{ind})$ (quadric-conic-structure) ,

$\mathfrak{F} = (X',X'',I_{ind})$ (point-quadric-structure) ,

where I_{ind} donotes in each case the incidence relation induced by R_1 (or R_2) in the obvious sense.

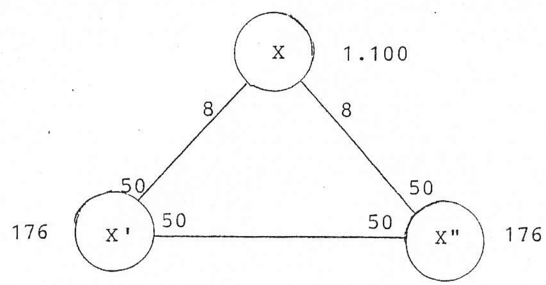
(4.10) THEOREM.

(1) $\mathfrak{F}'$ and $\mathfrak{F}''$ are 2-(176,8,2) designs, on which G acts as a group of automorphisms. $\bar{G}$ induces naturally an isomorphism between $\mathfrak{F}'$ and $\mathfrak{F}''$.

(2) $\mathfrak{F}$ is a symmetric 2-(176,50,14) design, on which G acts as a group of automorphisms. $\bar{G}$ acts on $\mathfrak{F}$ as a group of correlations interchanging points and quadrics.

(3) $(X',X'',X; R_1)$ provides for a model of G. Higman's geometry defined in [10]. G acts on this model as a group of automorphisms, $\bar{G}$ as a group of correlations interchanging points and quadrics and leaving the set of conics invariant.

Proof: Since G acts doubly-transitively on X' and X" and transitively on X we immediately obtain that $\mathfrak{F}'$ is a 2-(176,k', λ') design, that $\mathfrak{F}''$ is a 2-(176,k", λ'') design and that $\mathfrak{F}$ is a symmetric 2-(176,k, λ) design. From (4.8) it follows that $k' = k'' = 8$ and that $k = 50$. The canonical equations for the design parameters now yield $\lambda' = 2 = \lambda''$ and $\lambda = 14$. It is clear from the definition, that G acts on $\mathfrak{F}'$, $\mathfrak{F}''$ and $\mathfrak{F}$ as a group of automorphisms. It follows also from (4.8) that $\bar{G}$ induces an isomorphism between $\mathfrak{F}'$ and $\mathfrak{F}''$ and acts on $\mathfrak{F}$ as a group of correlations (inducing a polarity).
It is now straightforward to see that the "axioms" of G. Higman's geometry are fulfilled, see [10]. (Note that the mapping Θ of (4.7) is intimately related to the "conic correspondence" required in G. Higman's property (vi) in an obvious way.) □



(4.11) COROLLARY. G is isomorphic to the automorphism group of G. Higman's geometry. □

REMARK. G. Higman's natural correspondence between the unordered pairs of points and of quadrics is given by the mapping Θ of (4.7).

It is plainly clear that, as a general principle of construction, the additive structure of a linear code left invariant by a group G may be used to define incidence structures admitting G as a group of automorphisms.

5. Subgroups given by the code H_{23}

It has been shown by Conway [3] and Curtis [4] that the major part of the maximal subgroups of the Mathieu group M_{24} may be described in terms of the binary Golay code of length 24. In this section we will show that the code H_{23} serves for this purpose as well in the case of the Higman-Sims group.

We start with the following general concept.

(5.1) DEFINITION. Let a group G act on a set X.

(1) A subgroup U of G is called an X-<u>subgroup</u> if and only $U = G_x$ for some $x \in X$.

(2) The set of all X-subgroups of G is denoted by $sub_X(G)$. $sub_X(G)$ is a union of conjugacy classes.

(3) $sub_X(G)$ is partially ordered under inclusion. A subgroup U of G is called X-<u>maximal</u> if and only if $U \neq G$ and $U \leq V \in sub_X(G)$ implies $V \in \{U,G\}$.

The set of all X-maximal subgroups of G is denoted by $max_X(G)$.

Note that $max_X(G) = \emptyset$ if and only if G acts trivially on X.

In the following we retain the notation of the previous sections. In particular, G denotes the Higman-Sims group. We consider the action of G on the FG-module H_{23}.

(5.2) THEOREM. Every H_{23}-maximal subgroup of G is conjugate to one subgroup in the following list:

(a) $G_\alpha \cong M_{22}$ of index 100;

(b) $G_{\{\alpha,\beta\}} \cong P\Sigma L(3,4)$ of index 1.100 where α and β are joined in the Higman-Sims graph;

(c) $G_{\{\alpha,\gamma\}} = E_{16}\Sigma_6$ of index 3.850 where α and γ are not joined in the Higman-Sims graph;

(d) $G_{x_{30}} \cong \Sigma_8$ of index 1.100 where $x_{30} \in W_{30}(H_{23})$, a conic stabilizer in G. Higman's geometry;

(e) $G_{x_{36}} \cong 2^6\ GL(3,2)$ of index 4.125 where $x_{36} \in W_{36}(H_{23})$;

(f) $G_{x_{40}} \cong Z_2 \times P\Gamma L(2,9)$ of index 15.400 where $x_{40} \in 40_1$, the centralizer in G of a fixed-point-free involution;

(g') $G_{x'_{34}} \cong M_{11}$ of index 5.600 where $x'_{34} \in W_{34}(H'_{22})$;

(g") $G_{x''_{34}} \cong M_{11}$ of index 5.600 where $x''_{34} \in W_{34}(H''_{22})$;

(h') $G_{x'_{50}} \cong PSU(3,5^2)$ of index 352 where $x'_{50} \in X'_o$;

(h") $G_{x''_{50}} \cong PSU(3,5^2)$ of index 352 where $x''_{50} \in X''_o$.

Proof: The assertion of the theorem follows essentially from (3.11), (3.20) and its proof. Some arguments of the omitted proof of (3.13) are also required concerning the orbits in Ω of the stabilizers G_x, $x \in H_{22}$. Since these arguments are elementary and tedious we omit the details. □

Note that $\bar{G}$ = Aut(G) fuses the conjugacy classes (g') and (g"), resp. (h') and (h").

It can be shown that all H_{23}-maximal subgroups of G are in fact maximal subgroups of G with exception of (h') and (h"). The H_{23}-maximal subgroups of G of types (h') and (h") are contained with index 2 in maximal subgroups isomorphic to $P\Sigma U(3,5^2)$ as easily follows from section 4. From (3.21) we know that these maximal subgroups are H_{23}/H_1-groups. More precisely we have the following result.

(5.3) THEOREM. Every H_{23}/H_1-maximal subgroup of G is either H_{23}-maximal or a maximal subgroup of index 176 conjugate to a point-stabilizer or a quadric-stabilizer in G. Higman's geometry.

Proof: The assertion follows from (5.2), (3.20) and section 4. □

It ist not difficult to show that the subgroups of type (a), (b),... (g") are in fact maximal subgroups of G. Of course, this follows from Magliveras [18] where reference is given to his unpublished dissertation [17]. We give an example:

(5.4) LEMMA. Let $x \in W_{30}(H_{23})$. Then $G_x \cong \Sigma_8$ is a maximal subgroup of G.

Proof: Let $G_x < H \leq G$. G_x has just 2 orbits in Ω: supp $(x) = \Lambda_8(x)$ and $\Lambda_6(x)$ = supp $(\bar{x})$ where $\bar{x} = x + 1$, see (3.10). So we may conclude that H acts transitively on Ω.
It follows from (3.10) that $(G_x)_\beta$ has orbits of length 8 and 14 in $\Delta(\beta)$ if $\beta \in \Lambda_8(x)$ and that $(G_x)_\gamma$ has orbits of length 16 and 6 in $\Delta(\gamma)$ if $\gamma \in \Lambda_6(x)$. Moreover $(G_x)_\gamma$ contains a Sylow 7-subgroup of G for $\gamma \in \Lambda_6(x)$. We easily conclude that H_ξ acts primitively on $\Delta(\xi)$ for $\xi \in \Omega$. By a theorem of Wielandt [29, 31.1] H_ξ acts 2-transitively on $\Delta(\xi)$ (and it follows that H has the orbits $\{\xi\}$, $\Delta(\xi)$ and $\Delta \circ \Delta(\xi)$ in Ω). It follows from Conway [3] that $H_\xi = G_\xi$, since M_{22} has no proper subgroup acting doubly-transitively on 22 points. Hence H = G. □

It is a result of Magliveras [17;18] that G has only two conjugacy classes of maximal subgroups which are note H_{23}/H_1-maximal:

(i) The centralizer of an involution with fixed-point (induced by an elation in $PSL(3,4) = M_{21}$), of index 5.775, of structure $2^6\Sigma_5$, acting intransitively on Ω with two orbits of lengths 20 and 80.

(ii) The normalizer of the cyclic group generated by an element of order 5 whose centralizer is of order 300 in G, of index 36.960 and acting transitively on Ω with a system of imprimitivity of type 20^5.

It is easy to show that the groups in the class (i) are H_{78}-maximal subgroups of G. Note that H_{79} is the inverse image under v of $H_1 = \langle 1 \rangle$ and that $H_{22} = \mathrm{Im}\, v \leqslant H_{79}$; hence a forteriori every intransitive subgroup of G fixes a vector in $H_{79} \setminus H_1$.

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